# PML-FDTD in Cylindrical and Spherical Grids

F. L. Teixeira and W. C. Chew, *Fellow, IEEE*

***Abstract*— Perfectly matched layers (PML's) are derived for cylindrical and spherical finite-difference time-domain (FDTD) grids. The formulation relies on the complex coordinate stretching approach. Two-dimensional (2-D) cylindrical and three-dimensional (3-D) spherical staggered-grid FDTD codes are written based on the time-domain versions of the equations. Numerical simulations validate the formulation by showing very good agreement between the perfectly matched layer-finite-difference time-domain (FDTD) results and the free-space analytic solutions.**

## I. Introduction

The perfectly matched layer (PML) [1] proved to be a very efficient means to truncate the computational domain in the finite-difference time-domain (FDTD) method [2]–[6]. The original PML concept applied only to Cartesian coordinates. To extend its range of applicability, the PML concept was later extended to nonorthogonal FDTD grids [7], [8]. However, an approximate impedance matching condition was used, since the perfect matching condition was derived based on the assumption of the metric coefficients to be independent of the spatial coordinates.

In this work, we derive PML media for cylindrical and spherical coordinate systems with an exact formulation in the sense that it provides a reflectionless termination in the continuum limit. The formulation is based on the complex coordinate stretching approach [2]. Results are compared against analytic solutions.

## II. Formulation

With the following change of variables:

$$\tilde{\zeta} = \int_0^{\zeta} s_\zeta(\zeta')\, d\zeta' \tag{1}$$

where $s_\zeta(\zeta)$ are the complex stretching variables [2] and $\zeta$ stands for $x, y, z$, it is possible to show [9] that the modified Maxwell's Equations (ME's) [2] on a PML medium can be recast in the same form as the original ME's but on a complex variable spatial domain, $(\tilde{x}, \tilde{y}, \tilde{z})$. Closed-form solutions already obtained in the ordinary media can be mapped to the PML media through a simple analytic continuation of the spatial variables to a complex space. Moreover, this analytic continuation, if causal, can be easily generalized to other coordinate systems to provide PML's on these systems [9].

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The authors are with the Center for Computational Electromagnetics, Electromagnetics Laboratory, Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801-2991 USA.

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*1) Cylindrical PML:* The PML formulation for a cylindrical coordinate system proceeds by writing the ME's on a complex cylindrical coordinate. Only the TM$_z$ case will be addressed. The TE$_z$ case follows by duality. Since in the $z$ direction the PML formulation does not change, only the 2-D problem is treated. To achieve the reflectionless absorption of the outward traveling waves, the radial coordinate is mapped through ($e^{-i\omega t}$ convention)

$$\begin{aligned}\rho \rightarrow \tilde{\rho} &= \int_0^{\rho} s_\rho(\rho')\, d\rho' = \int_0^{\rho} \left(a_\rho(\rho') + i\frac{\sigma_\rho(\rho')}{\omega}\right) d\rho' \\ &= b_\rho(\rho) + i\frac{\Delta_\rho(\rho)}{\omega}.\end{aligned} \tag{2}$$

$a_\rho$ and $\sigma_\rho$ are added degrees of freedom. By splitting the $z$-component of the electric field $E_z = E_{z\rho} + E_{z\phi}$, ME's in complex space are cast in a form suitable for time-stepping:

$$(i\omega s_\rho)\epsilon \tilde{E}_{z\rho} = -\frac{\partial}{\partial \rho}(\tilde{\rho} H_\phi) \tag{3a}$$

$$(i\omega \tilde{\rho}) E_{z\rho} = i\omega \tilde{E}_{z\rho} \tag{3b}$$

$$(i\omega \tilde{\rho})\epsilon E_{z\phi} = \frac{\partial H_\rho}{\partial \phi} \tag{3c}$$

$$(i\omega s_\rho)\mu H_\phi = -\frac{\partial}{\partial \rho}(E_{z\rho} + E_{z\phi}) \tag{3d}$$

$$(i\omega \tilde{\rho})\mu H_\rho = \frac{\partial}{\partial \phi}(E_{z\rho} + E_{z\phi}). \tag{3e}$$

Only the $E_z$ component ($H_z$ in the TE$_z$ case) is split. This is because the transversal problem has stretching only on the $\rho$ direction. This is in contrast to the Cartesian case where all field components need to be split in the split-field formulation [1], [3]. However, additional fields $\tilde{E}_{z\rho}$ and $\tilde{\rho} H_\phi$ are needed. The time-domain version of (3) is implemented in the usual cylindrical staggered-grid scheme [12].

Alternative generalizations of the PML to cylindrical coordinates were also recently considered in [10] and [11]. The approach of [10] is based on the anisotropic formulation [5], [6], and the approach of [11] is based on a modified version of ME's amenable to be recast on a well-posed scheme.

*2) Spherical PML:* In spherical coordinates, the analytic continuation is on the radial variable $r$:

$$\begin{aligned}r \rightarrow \tilde{r} &= \int_0^{r} s_r(r')\, dr' = \int_0^{r} \left(a_r(r') + i\frac{\sigma_r(r')}{\omega}\right) dr' \\ &= b_r(r) + i\frac{\Delta_r(r)}{\omega}\end{aligned} \tag{4}$$

from which the modified Faraday's law in a form suitable for time-stepping reads:

$$(i\omega \tilde{r})\mu H_r = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta E_\phi) - \frac{\partial E_\theta}{\partial \phi}\right] \tag{5a}$$

$$(i\omega s_r)\mu\tilde{H}_\theta = \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(s_r E_r) - \frac{\partial\tilde{E}_\phi}{\partial r} \tag{5b}$$

$$(i\omega s_r)\mu\tilde{H}_\phi = \frac{\partial\tilde{E}_\theta}{\partial r} - \frac{\partial}{\partial\theta}(s_r E_r) \tag{5c}$$

$$(i\omega\tilde{r})H_\theta = i\omega\tilde{H}_\theta \tag{5d}$$

$$(i\omega\tilde{r})H_\phi = i\omega\tilde{H}_\phi. \tag{5e}$$

The modified Ampere's law follows by duality. In spherical coordinates there is no need to split the fields at all, since the PML is achieved through complex stretching on the radial variable, $r$, only. This is in contrast to the three-dimensional (3-D) Cartesian case, where there are 12 field components after field splitting and six boundary surfaces to treat. However, additional fields components are needed inside the PML: $s_r E_r$, $\tilde{E}_\theta$, $\tilde{E}_\phi$, $s_r H_r$, $\tilde{H}_\theta$, and $\tilde{H}_\phi$. The spatial discretization scheme and the treatment of the singularities on the spherical grid follows [13].

## III. Numerical Results

The formulation is validated against analytic solutions obtained by solving free-space problems in the frequency domain, multiplying by the source pulse spectrum, and inverse Fourier-transforming. In both examples, a quadratic taper on $\sigma_\rho$ and $\sigma_r$ is used inside the PML and, for simplicity, $a_\rho = a_r = 1$ everywhere.

Fig. 1 shows the normalized $E_z$ field computed using the analytic formulation and the 2-D FDTD algorithm for a line source in a cylindrical grid. The excitation is the derivative of a Blackman–Harris pulse centered at $f_c = 300$ MHz. The line source is at $(r, \phi) = (7.5\lambda_c, 0°)$ and the field is sampled at $(r, \phi) = (6.5\lambda_c, 0°)$, where $\lambda_c = c/f_c$. The grid has a hard termination at $r = 9\lambda_c$. The FDTD algorithm includes a eight-layer cylindrical PML region before the grid ends. The PML thickness is $0.5\lambda_c$, for a cell size $\Delta\rho = \lambda_c/16$ in the radial direction. The curves are in excellent agreement and no reflection is visible. To illustrate the high absorption achieved, the inset shows the simulation of the same problem without the PML. In the PML-FDTD simulation, the maximum residual amplitude of the normalized $E_z$ field over the time-window of the reflected pulse is less than $5 \times 10^{-3}$, which is less than 0.8% of the maximum amplitude present in the simulation without PML. The residual field can be attributed not only to spurious reflections but also to numerical dispersion effects.

Fig. 2 shows the normalized $E_\theta$ field computed with the analytic formulation and the 3-D FDTD algorithm for a point source in a spherical grid with the same excitation pulse. The $\theta$-polarized dipole is at $(r, \theta, \phi) = (2.5\lambda_c, 90°, 0°)$. The field is sampled at $(r, \theta, \phi) = (3\lambda_c, 90°, 0°)$. The grid is terminated at $r = 4\lambda_c$. The FDTD algorithm includes a eight-layer spherical PML region before the grid ends. The PML thickness is $0.8\lambda_c$, for a cell size $\Delta r = \lambda_c/10$ in the radial direction. Again, no reflection is visible. The small oscillation after the passage of the incident pulse is also present in the simulation without PML and can be attributed to the discretization errors and numerical dispersion effects due to the high grid curvature in the simulation region. For illustration, the inset shows the simulation of the same problem without the PML. In the PML-FDTD simulation, the maximum residual amplitude of the normalized $E_z$ field for $t > 9$ ns is less than $1 \times 10^{-2}$, which is less than 3% of the maximum amplitude present in the simulation without PML.

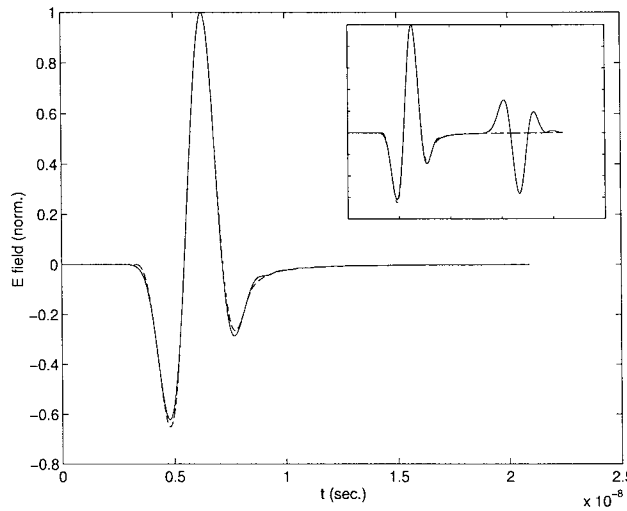


Fig. 1. Analytic solution for a line source on free-space (dashed line) versus 2-D cylindrical-grid FDTD solution with eight-layer cylindrical PML (solid line). The inset illustrates the result of the simulation without the PML.

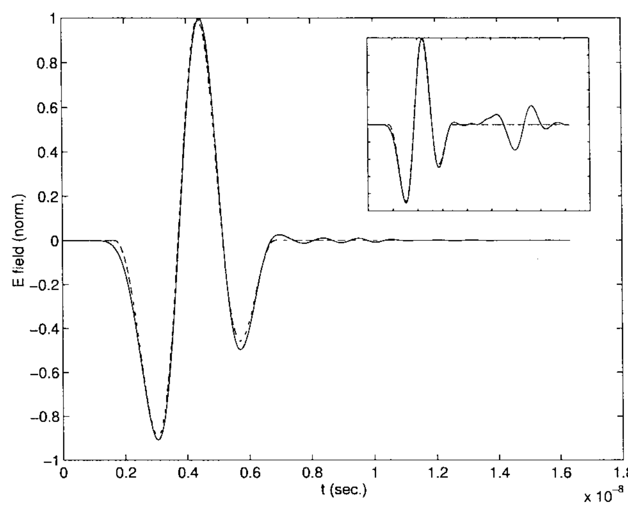


Fig. 2. Analytic solution for an infinitesimal electric dipole on free-space (dashed line) versus 3-D spherical-grid FDTD solution with eight-layer spherical PML (solid line). The inset illustrates the result of the simulation without the PML.

## IV. Conclusion

PML's are derived for cylindrical and spherical coordinates. The formulation is based on the complex coordinate stretching approach. FDTD codes are developed based on the time-domain versions of the equations. The accuracy of the formulation is validated by computing the field radiated from line and point sources on cylindrical and spherical grids against known analytic results.

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## REFERENCES


[1] J. Berenger, "A perfectly matched layer for the absorption of electromagnetic waves," *J. Comput. Phys.*, vol. 114, pp. 185–200, 1994.

[2] W. C. Chew and W. Weedon "A 3D perfectly matched medium from modified Maxwell's equations with stretched coordinates," *Microwave Opt. Tech. Lett.*, vol. 7, no. 13, pp. 599–604, 1994.

[3] D. S. Katz, E. T. Thiele, and A. Taflove, "Validation and extension to three dimensions of the Berenger PML absorbing boundary condition," *IEEE Microwave Guided Wave Lett.*, vol. 4, pp. 268–270, Aug. 1994.

[4] C. M. Rappaport, "Interpreting and improving the PML absorbing boundary condition using anisotropic lossy mapping of space," *IEEE Trans. Magn.*, vol. 32, no. 3, pp. 968–974, 1996.

[5] Z. S. Sacks, D. M. Kingsland, R. Lee, and J.-F. Lee, "A perfectly matched anisotropic absorber for use as an absorbing boundary condition," *IEEE Trans. Antennas Propagat.*, vol. 43, no. 12, pp. 1460–1463, 1995.

[6] S. D. Gedney, "An anisotropic perfectly matched layer absorbing medium for the truncation of FDTD lattices," *IEEE Trans. Antennas Propagat.*, vol. 44, pp. 1630–1639, Dec. 1996.

[7] E. A. Navarro, C. Wu, P. Y. Chung, and J. Litva, "Application of PML super-absorbing boundary condition to nonorthogonal FDTD method," *Electron. Lett.*, vol. 30, no. 20, pp. 1654–1655, 1994.

[8] J. A. Roden and S. D. Gedney, "Efficient implementation of the uniaxial-based PML media in three-dimensional nonorthogonal coordinates with the use of the FDTD technique," *Microwave Opt. Tech. Lett.*, vol. 14, no. 2, pp. 71–75, 1997.

[9] W. C. Chew, J. M. Jin, and E. Michielssen, "Complex coordinate system as a generalized absorbing boundary condition," in *Proc. 13th Annu. Rev. of Prog. Appl. Comp. Eletromag.*, Monterey, CA, Mar. 17–21, 1997, vol. 2, pp. 909–914.

[10] J. Maloney, M. Kesler, and G. Smith, "Generalization of PML to cylindrical geometries," in *Proc. 13th Annu. Rev. Prog. Appl. Comp. Eletromag.*, Monterey, CA, Mar. 17–21, 1997, vol. 2, pp. 900–908.

[11] B. Yang, D. Gottlieb, and J. S. Hesthaven, "On the use of PML ABC's in spectral time-domain simulations of electromagnetic scattering," in *Proc. 13th Annu. Rev. of Prog. Appl. Comp. Eletromag.*, Monterey, CA, Mar. 17–21, 1997, vol. 2, pp. 926–933.

[12] M. Fusco, "FDTD algorithm in curvilinear coordinates," *IEEE Trans. Antennas Propagat.*, vol. 38, pp. 76–89, Jan. 1990.

[13] R. Holland, "THREDS: A finite-difference time-domain EMP code in 3D spherical coordinates," *IEEE Trans. Nucl. Sci.*, vol. 30, pp. 4592–4595, 1983.